CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ as diameters, circles are described passing through the pole. Find the locus of their points of intersection, and show that the area is twice that of the loop.

SOLUTION BY THEODORE HOWARD, New Haven, Conn.

Let ϕ be the angle between the polar axis and the radius vector of the circle described on the radius vector of the lemniscate as a diameter. Let r be the radius vector of the circle, the origin and polar axis being the same as for the lemniscate. Then the equation of the circle is $r = \rho \cos(\phi - \theta)$. Or $r^2 = \rho^2 \cos^2(\phi - \theta) = a^2 \cos 2\theta \cos^2(\phi - \theta)$, substituting from the equation of the lemniscate. In this equation, θ is the variable parameter. Taking the derivative of this equation with respect to θ, we have

$$\begin{aligned} 0 &= 2a^2 \cos(\phi - \theta)[\sin(\phi - \theta) \cos 2\theta - \sin 2\theta \cos(\phi - \theta)], \\ &= 2a^2 \cos(\phi - \theta) \sin(\phi - 3\theta). \end{aligned}$$

Hence, $\theta = \frac{\phi}{3}$, or $\theta = \phi - \frac{\pi}{2}$ an extraneous value. Substituting $\frac{\phi}{3}$ for θ in the preceding equation, we have $r^2 = a^2 \cos^3 \frac{2\phi}{3}$, the equation of the envelope. The area of one loop of the envelope is $A = \frac{a^2}{2}\int_b^a \cos^3 \frac{2\phi}{3} d\phi = 3a^2 \int_0^{\pi/4} \cos^3 2\theta d\theta = a^2$. The area of one loop of the lemniscate is

$$A' = a^2 \int_0^{\pi/4} \cos 2\theta d\theta = \frac{a^2}{2}.$$

Hence,

$$A = 2A'.$$

Also solved by C. E. HORNE, PAUL CAPRON, H. L. AGARD, and NORMAN ANNING.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

SOLUTION BY W. W. BEMAN, University of Michigan.

Assume $z = \phi(y + mx)$. The equation in m is $2m^2 - 3m - 2 = 0$. Hence,

$$z = \phi(y + 2x) + \psi(2y - x).$$

Also solved by ELIJAH SWIFT, J. L. RILEY, and the PROPOSER.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

SOLUTION BY H. L. AGARD, Williams College.

The rope leaves the post in such a manner that the knot at the end of the noose is in front of the middle of one face of the post. Let a be the thickness of the post, φ the angle at which the rope leaves the post, b the distance from the knot to the corner of the post, and c the perpendicular distance from the knot to the post.

Then

$$b = \frac{a}{2}\sec\varphi \qquad \text{and} \qquad c = \frac{a}{2}\tan\varphi.$$